

# Coulomb Gas on the Half Plane<sup>★</sup>

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## Abstract

The Coulomb-gas description of minimal models is considered on the half plane. Screening prescriptions are developed by the perturbative expansion of the Liouville theory with imaginary coupling and with Neumann boundary condition on the bosonic field. To generate the conformal blocks of more general boundary conditions, we propose the insertion of boundary operations.

*Key words:* Boundary conformal field theory, Coulomb gas, screening operators, boundary conditions, Ising model.

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## 1 Introduction

The investigation of boundary quantum integrable models [1,2] is motivated both by the abundance of applications, such as the Kondo effect [3], open string theory and quantum wires [4], as well as by the clear description of their bulk counterparts as perturbations around conformal field theories [5]. In the bulk, Toda field theory [6,7], free field theory and Coulomb-gas description [8–10] are most useful tools. In the presence of boundaries, the conformal theory is well understood [11–14]; for systems off criticality however, integrability sometimes only allows certain combinations of bulk and boundary terms [15,16,2].

The Coulomb gas plays a special role in this game. Being a (non-affine) Toda field theory, it carries most of the features of the off-critical systems. On the other hand, it serves perfectly well to describe minimal conformal models. In

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this paper, we wish to consider the Coulomb-gas description of minimal conformal models on the half plane, and to find connections between its boundary conditions and screening contours.

In section 2, we will first consider the conformal invariance of the Toda action. For the half plane, this leads to two conformally invariant boundary conditions on the free field. One of them is the Neumann boundary condition, for which the discussion looks most natural.

Section 3 is a brief review of the Coulomb-gas formulation of minimal conformal models. The Coulomb gas is treated as a Liouville theory with imaginary coupling, which is expanded as a marginal perturbation around a free bosonic field theory. This description with two-dimensional screening integrals was introduced in [8] as a manifestly conformally invariant alternative. We will compare it to the contour-integral description, following the lines of Mathur [10] who considered the full plane.

The Ising model is considered as an example in section 4. We will show to what contour integrals the two-point functions reduce in the above mentioned description with Neumann boundary condition on the bosonic field. In section 4.2, these results are compared to the conformal blocks of the free or fixed boundary conditions on the spin operator [11,17] which both require boundary-crossing integration contours.

This motivates the introduction of new boundary terms in section 5 which are best described as composite operators of a vertex operator and its mirror image. The expressions introduced are shown to lead either to vanishing correlators or to sew together screening contours of the two different half planes, leaving the vertex operator corresponding to an identity operator of the minimal model at the boundary. Correspondences to the boundary states in [12] are discussed.

The outlook in section 6 compares our boundary expressions to the boundary terms added to the off-critical action in [1] and to the boundary terms in Toda theories [15,16]. Section 7 summarises the paper.

In the Appendix, it is shown that the two-dimensional integral fulfills the Ward identities as well for the half plane, and how the corresponding contour integrals can be derived and evaluated.

## 2 Conformal invariance of the Toda action

Consider, following the approach of [6], the Toda action

$$S_{\text{bulk}}^{\text{class.}} = \frac{1}{8\pi} \iint_M d^2z \sqrt{g} \left[ g^{ab} (\partial_a \Phi) \cdot (\partial_b \Phi) - \frac{2}{\beta^2} \sum^r e^{\beta \alpha_i \cdot \Phi} + \frac{2}{\beta} R \varrho \cdot \Phi \right], \quad (2.1)$$

where  $g_{ab}$  is the metric of the two-dimensional manifold  $M$  with curvature  $R$ . The  $\alpha_i$  are the simple roots of the ordinary Lie algebra  $\mathfrak{g}$  with rank  $r$ . Its Weyl vector is  $\varrho = \sum^r \lambda_i$ , the fundamental weights  $\lambda_i$  are defined by  $\lambda_i \cdot \alpha_j = \delta_{ij}$ . The coupling constant  $\beta$  eventually will be sent to  $\beta \rightarrow i\tilde{\beta}$ , with  $\tilde{\beta}$  real.

The energy-momentum tensor  $T_{ab}^{\text{class.}} \equiv -\frac{4\pi}{\sqrt{g}} \frac{\delta S^{\text{class.}}}{\delta g^{ab}}$  is traceless on a flat world sheet ( $R = 0$ ) where it simplifies with the help of the equations of motion to a holomorphic  $T(z)$  and an antiholomorphic  $\bar{T}(\bar{z})$ :

$$\begin{aligned} T_{zz}^{\text{class.}} &= -\frac{1}{2}(\partial_z \Phi)^2 + \frac{1}{\beta} \varrho \cdot (\partial_z^2 \Phi) \equiv T(z), \\ T_{\bar{z}\bar{z}}^{\text{class.}} &= -\frac{1}{2}(\partial_{\bar{z}} \Phi)^2 + \frac{1}{\beta} \varrho \cdot (\partial_{\bar{z}}^2 \Phi) \equiv \bar{T}(\bar{z}), \\ T_{z\bar{z}}^{\text{class.}} &= T_{\bar{z}z}^{\text{class.}} = 0. \end{aligned} \quad (2.2)$$

Under the transformation

$$\begin{aligned} g_{ab} &\rightarrow \Omega^2 g_{ab}, \quad g^{ab} \rightarrow \Omega^{-2} g^{ab}, \quad \sqrt{g} \rightarrow \Omega^2 \sqrt{g}, \\ \Phi &\rightarrow \Phi - \frac{1}{\beta} \ln \Omega \varrho, \quad R \rightarrow \Omega^{-2} [R - 2\nabla_a \nabla^a \ln \Omega], \end{aligned} \quad (2.3)$$

the action (2.1) changes to

$$\begin{aligned} S^{\text{class.}} &\rightarrow S^{\text{class.}} - \frac{1}{8\pi} \iint_M d^2z \sqrt{g} \frac{4\varrho^2}{\beta^2} \ln \Omega (R - \nabla_a \nabla^a \ln \Omega) - \\ &\quad - \frac{1}{8\pi} \int_{\partial M} dx \sqrt{g} \frac{4}{\beta} (\partial^\perp \ln \Omega) \varrho \cdot \left( \Phi - \frac{1}{\beta} \varrho \ln \Omega \right), \end{aligned} \quad (2.4)$$

where we have included the possibility of a boundary. The role of the curvature term  $R \varrho \cdot \Phi$  in the action (2.1) is to make the change of the bulk term independent of  $\Phi$ .

In the absence of boundaries and for vanishing curvature, the action is conformally invariant if

$$\partial_z \partial_{\bar{z}} \ln \Omega = 0 \quad \Rightarrow \quad \Omega = g(z) h(\bar{z}), \quad (2.5)$$

i.e. conformal invariance restricts to analytic conformal transformations. Thus the conformal freedom is equivalent to general holomorphic changes of  $z$ , and the model has two independent Virasoro symmetries, corresponding to  $g$  and  $h$  in (2.5). The flatness of the world sheet is preserved under such transformations.

On the quantum level, normal ordered expressions appear in the action, the equations of motion, and the energy-momentum tensor [18,19]. As a consequence, the energy-momentum tensor (2.2) is not conserved, and the conformal invariance is broken. To correct this, we have to change the prefactor of the curvature term in (2.1) from  $\frac{2}{\beta}$  to  $2(\beta + \frac{1}{\beta})$ . This change does not affect the boundary term in (2.4), the conformal invariance of the action is assured by (2.5). After having applied the quantum equations of motion

$$\partial_z \partial_{\bar{z}} \Phi = -\frac{1}{\beta} \sum^r \alpha_i :e^{\beta \alpha_i \cdot \Phi}: m^{-2\beta^2} , \quad (2.6)$$

where  $m$  is a regulator which eventually can be set to zero, the energy-momentum tensor reads [6]

$$T_{zz} = -\frac{1}{2} :(\partial_z \Phi)^2: + (\beta + \frac{1}{\beta}) \varrho \cdot (\partial_z^2 \Phi) , \quad (2.7)$$

and similar for  $\bar{T}$ . The conformal anomaly of this energy-momentum tensor is [6]

$$c = r + 12\varrho^2 \left[ \beta + \frac{1}{\beta} \right]^2 . \quad (2.8)$$

## 2.1 Toda theory on the half plane

Now consider the same models on the half plane  $y > 0$ , where  $z = x + iy$  and the real axis is the boundary  $\partial M$ , and take into account the boundary term in (2.4), as well. This term still contains  $\Phi$ , so it asks for a boundary term in the action which will cancel the  $\Phi$ -dependence. One can modify (2.1) to

$$S = S_{\text{bulk}} - \frac{1}{8\pi} \int_{\partial M} dx \frac{8}{\beta} K \varrho \cdot \Phi . \quad (2.9)$$

$K(x) = \Gamma_{xx}^y|_{y=0}$  is the extrinsic curvature of the boundary. The variation of  $K$  under (2.3) close to the flat metric is  $\delta K = \frac{1}{2}(\partial - \bar{\partial}) \ln \Omega = -\frac{1}{2i} \partial_y \ln \Omega$ . Hence, the change of the boundary term in (2.9) cancels the  $\Phi$ -dependent boundary term in (2.4), leaving us with the additional  $\Phi$ -independent conditions

$$\partial_y \ln \Omega|_{y=0} \equiv i(\partial - \bar{\partial}) \ln \Omega|_{y=0} = 0 \quad \text{and} \quad K = 0 . \quad (2.10)$$

Since the variation of  $K$  is proportional to (2.10) the boundary term in (2.9) does not affect the energy-momentum tensor  $T \equiv -\frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g}$ .

The second condition in (2.10) is fulfilled trivially for the geometry chosen. The first condition in (2.10) restricts  $g$  and  $h$  from (2.5) to be the analytic continuation of each other. This implies that the antiholomorphic part of the energy-momentum tensor has to be the analytic continuation of its holomorphic part, leaving us with a single copy of the Virasoro algebra [11]. In other words, the component  $T_{xy}$  of the energy-momentum tensor has to vanish on the real axis [20]

$$T_{xy} = 0, \quad \text{at } y = 0. \quad (2.11)$$

Using the normal-ordering procedure of [19], one finds that

$$\begin{aligned} T_{xy} &= \frac{1}{4i}(T_{zz} + T_{\bar{z}\bar{z}} - T_{z\bar{z}} - T_{\bar{z}z}) = \frac{1}{4i}(T_{zz} - T_{\bar{z}\bar{z}}) \\ &= \begin{cases} \frac{1}{8}(\partial_x \Phi) \cdot (\partial_y \Phi) - \frac{1}{4\beta} \varrho \cdot \partial_x \partial_y \Phi & \text{classically,} \\ \left\{ \begin{aligned} &\frac{1}{16} \left[ :(\partial_x \Phi) \cdot (\partial_y \Phi): + :(\partial_y \Phi) \cdot (\partial_x \Phi): \right] - \\ &-\frac{1}{4}(\beta + \frac{1}{\beta}) \varrho \cdot \partial_x \partial_y \Phi \end{aligned} \right\} & \text{quantum,} \end{cases} \end{aligned} \quad (2.12)$$

and (2.11) is fulfilled if one takes the Neumann boundary condition

$$\partial_y \Phi|_{y=0} = 0. \quad (2.13)$$

It can also be fulfilled by taking<sup>2</sup>

$$\partial_y \Phi|_{y=0} = \begin{cases} \sum \alpha_i e^{\frac{\beta}{2} \alpha_i \Phi} & \text{classically [15],} \\ \sum \alpha_i :e^{\gamma \alpha_i \Phi}: m^{-2\gamma^2} & \text{quantum,} \end{cases} \quad (2.14)$$

where  $\gamma = \frac{1}{2}\beta$  or  $\gamma = \frac{1}{2}\frac{1}{\beta}$ . Both conditions (2.13) and (2.14) are conformally invariant, since – using that at the boundary  $\varphi(z) = \tilde{\varphi}(\bar{z})$  – the right hand sides of (2.14) have dimension 1. For the non-affine  $A_1$  (Coulomb gas) which we will consider throughout the rest of the paper, the two possibilities for  $\gamma$  correspond to the screening operators  $V_{\alpha_-}$  and  $V_{\alpha_+}$  which are the natural dimension-one objects one can put on the right hand side of (2.14). The same two choices exist as well for all other simply-laced Lie algebras. For affine algebras, only  $\gamma = \frac{1}{2}\beta$  has been considered so far [16].

For the Dirichlet boundary condition  $\Phi|_{y=0} = 0$ , the term  $\partial_x \Phi|_{y=0}$  would vanish everywhere on the boundary, but the term  $\partial_y \partial_x \Phi$  generically would not. Thus

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<sup>2</sup> We restrict our consideration to simply-laced Lie algebras.

the Dirichlet boundary condition does not naturally fit into our description. For the sine-Gordon model, the UV limit of the appropriate version of (2.14) is the Neumann boundary condition (2.13), while the IR limit is the Dirichlet boundary condition  $\Phi|_{y=0} = \Phi_0$  [21].

In the following, we will choose Neumann boundary conditions for the bosonic field, i.e. we will work at the UV fixed point.

### 3 The Coulomb-gas description of minimal models

After this excursion to Toda field theory, we can try to apply the results to the Coulomb-gas representation. We will restrict the discussion to the non-affine algebra  $A_1$ , and consider the Liouville theory as a marginal perturbation of a free bosonic field theory. On the half plane, the bosonic field has to fulfill the Neumann boundary condition.

Consider the free bosonic field theory  $\mathcal{A} = \frac{1}{8\pi} \iint d^2z \left[ (\partial\Phi)(\bar{\partial}\Phi) + 2\sqrt{2}i\alpha_0 R\Phi \right]$ . Using  $r = 1$ ,  $\varrho^2 = \frac{1}{2}$  and  $\tilde{\beta} = -\sqrt{\frac{M}{M+1}}$ , think of the Liouville potential in (2.1) as a perturbation

$$\mathcal{A}_{\text{pert.}} = \frac{1}{4\pi\alpha_-^2} \iint d^2z :e^{\sqrt{2}i\alpha_- \Phi(z,\bar{z})}: . \quad (3.1)$$

The charges  $\alpha_0 = \frac{1}{\sqrt{4M(M+1)}}$ ,  $\alpha_{\pm} = \alpha_0 \pm \sqrt{\alpha_0^2 + 1}$  and  $\alpha_{n,m} = \frac{1-n}{2}\alpha_+ + \frac{1-m}{2}\alpha_-$  should not be confused with the simple roots  $\alpha_i$  from above. The energy-momentum tensor  $T_{zz} = -\frac{1}{2} :(\partial_z\Phi)^2: + \sqrt{2}i\alpha_0 (\partial_z^2\Phi)$  has the conformal anomaly  $c = 1 - 24\alpha_0^2 = 1 - \frac{6}{M(M+1)}$ . The vertex operators  $V_{\alpha_{n,m}} = :e^{\sqrt{2}i\alpha_{n,m}\varphi}:$  carry the charges and conformal weights of the Kac table [8].

The perturbation (3.1) changes the correlation functions to

$$\begin{aligned} \langle X \rangle_{\mathcal{A}+\mathcal{A}_{\text{pert.}}} &= \\ &= \langle e^{\frac{1}{4\pi\alpha_-^2} \iint d^2z :e^{\sqrt{2}i\alpha_- \Phi(z,\bar{z})}:} X \rangle_{\mathcal{A}} \\ &= \langle X \rangle_{\mathcal{A}} + \frac{1}{4\pi\alpha_-^2} \iint d^2z \langle :e^{\sqrt{2}i\alpha_- \Phi(z,\bar{z})}: X \rangle_{\mathcal{A}} + \\ &\quad + \frac{1}{16\pi^2\alpha_-^4} \iint d^2z \iint d^2w \langle :e^{\sqrt{2}i\alpha_- \Phi(z,\bar{z})}: :e^{\sqrt{2}i\alpha_- \Phi(w,\bar{w})}: X \rangle_{\mathcal{A}} + \dots \end{aligned} \quad (3.2)$$

for an arbitrary insertion  $X$ . Charge conservation restricts this expansion to

only one term<sup>3</sup>. The result is the Coulomb-gas formulation of minimal models with manifestly monodromy-invariant combinations of the holomorphic and antiholomorphic sectors. This was introduced in [22] as an alternative to the contour integrals. In [23,10], Stoke's theorem was used to show that this description coincides with the contour-integral description.

The non-vanishing contribution in (3.2) of the two-point function with insertion  $X = V_{\alpha_{12}}(z_I)V_{\alpha_{12}}(\bar{z}_I)V_{\alpha_{12}}(z_{II})V_{\alpha_{12}}(\bar{z}_{II})$  is for example

$$\begin{aligned}
& k \iint d^2z \langle V_{\alpha_-}(z)V_{\alpha_-}(\bar{z})V_{\alpha_{1,2}}(z_I)V_{\alpha_{1,2}}(\bar{z}_I)V_{\alpha_{1,2}}(z_{II})V_{\alpha_{1,2}}(\bar{z}_{II}) \rangle \\
&= k \iint d^2z \partial_{\bar{z}} \int_{\bar{z}}^{\bar{z}} dt \langle V_{\alpha_-}(z)V_{\alpha_-}(t)V_{\alpha_{1,2}}(z_I)V_{\alpha_{1,2}}(\bar{z}_I)V_{\alpha_{1,2}}(z_{II})V_{\alpha_{1,2}}(\bar{z}_{II}) \rangle \\
&= \frac{-i}{2}k \int_{\partial M} dz \int_{z^*}^z dt \langle V_{\alpha_-}(z)V_{\alpha_-}(t)V_{\alpha_{1,2}}(z_I)V_{\alpha_{1,2}}(\bar{z}_I)V_{\alpha_{1,2}}(z_{II})V_{\alpha_{1,2}}(\bar{z}_{II}) \rangle . \quad (3.3)
\end{aligned}$$

For the full plane, the integrand splits into a product of a holomorphic and an antiholomorphic factor<sup>4</sup>. Mathur [10] treats the branch cuts as the boundary  $\partial M$  along which Stoke's theorem has to be applied. The  $t$ -integration splits into contours between two singular points and the so-called  $J$ -terms, which go from a singularity to the complex conjugate  $z^*$  of the other integration variable. The  $J$ -terms vanish for monodromy reasons. Hence, Mathur is left with products of holomorphic and antiholomorphic block functions.

For the half plane, the Neumann boundary condition implies the correlator to be  $\langle \varphi(z)\tilde{\varphi}(\bar{w}) \rangle = -\ln(z - \bar{w})$  and  $\langle \tilde{\varphi}(\bar{z})\varphi(w) \rangle = -\ln(\bar{z} - w)$ , and the integrand in (3.3) will not split into two sectors<sup>5</sup>. The integral (3.3) is hence

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<sup>3</sup> We will abbreviate the coefficient  $\frac{1}{4\pi\alpha_-^2}$  with  $k$ . It will later be important, when we have to find the relative coefficient of conformal blocks with a different number of screeners. Since we treat the Liouville potential as a marginal perturbation,  $k$  is actually a free parameter which can be chosen arbitrarily.

<sup>4</sup> Recall that on the full plane the two sectors of the free field  $\Phi(z, \bar{z}) = \varphi(z) + \tilde{\varphi}(\bar{z})$  have a trivial contraction  $\langle \varphi\tilde{\varphi} \rangle = 0$ , and the vertex operator is

$$:e^{\sqrt{2}i\alpha_- \Phi(z, \bar{z})}: \equiv :e^{\sqrt{2}i\alpha_- \varphi(z)}: :e^{\sqrt{2}i\alpha_- \tilde{\varphi}(\bar{z})}: . \quad (3.4)$$

Our vertex operators  $V_{\alpha_{n,m}}$  and screening operators  $V_{\alpha_{\pm}}$  are meant to be either the holomorphic or the antiholomorphic part of such a splitting.

<sup>5</sup> For the half plane, we have

$$\begin{aligned}
:e^{\sqrt{2}i\alpha_- \Phi(z, \bar{z})}: &= (-1)^{-\alpha_-^2} :e^{\sqrt{2}i\alpha_- \varphi(z)}: :e^{\sqrt{2}i\alpha_- \tilde{\varphi}(\bar{z})}: \\
&= (-1)^{+\alpha_-^2} :e^{\sqrt{2}i\alpha_- \tilde{\varphi}(\bar{z})}: :e^{\sqrt{2}i\alpha_- \varphi(z)}: . \quad (3.5)
\end{aligned}$$

Reality is ensured by setting  $\langle :e^{\sqrt{2}i\alpha_- \Phi(z, \bar{z})}: \rangle = |(z - \bar{z})^{\alpha_-^2}|^2 = (z - \bar{z})^{\alpha_-^2} (\bar{z} - z)^{\alpha_-^2}$

a double integral of two screeners around four points. This is not surprising since the two-point function on the half plane has to fulfill the same differential equation as the holomorphic sector of a four-point function on the full plane [11]. Note, however, that this integral is a double-valued function. Uniqueness of the full-plane four-point function is obtained by combining the holomorphic and antiholomorphic blocks in a monodromy-invariant way. For the half-plane two-point function, both branches of the double-valued function are invariant under a twist of the two points in the upper half plane around each other and a simultaneous twist of their mirror images. The restriction to a unique function is obtained by observing that the  $z$ -integration has to stay in the upper half plane, while the  $t$ -integration is performed in the lower half plane.

In a similar way, one has to use for a generic  $N$ -point function on the half plane that one cannot get contours which go from one half plane to the other. Therefore, one gets considerably fewer conformal blocks than for one sector of the  $2N$ -point function on the full plane. Invariance under twists of the points  $(z_i - z_j) \rightarrow e^{2\pi i} (z_i - z_j)$  and simultaneous twists of their mirror images  $(\bar{z}_i - \bar{z}_j) \rightarrow e^{-2\pi i} (\bar{z}_i - \bar{z}_j)$  restricts then to unique results.

## 4 Example: The Ising model on the half plane

In this section and in the Appendix, we want to illustrate what the boundary  $\partial M$  will look like for the half plane, and with what tools one can evaluate the integral (3.3).

As an example, consider the Ising model on the half plane. Special care should be taken of the difference between the Neumann boundary condition on the free field  $\Phi$ , which arises naturally in the Coulomb-gas description, and the free and fixed boundary conditions on the  $\sigma$  operator [11]. Throughout the rest of this paper, it will be understood that the boundary condition on  $\Phi$  is the Neumann boundary condition. Superscripts “free” and “fixed” will refer to the boundary condition on  $\sigma$ .

Label the points by  $z_1, z_2, z_3, z_4, \dots$  or by  $z_I, \bar{z}_I, z_{II}, \bar{z}_{II}, \dots$  depending on whether there is an emphasis on the properties of the holomorphic block on the full plane or of the  $N$ -point function on the half plane. See Figure 1 for an illustration.

Define the cross ratio to be

$$\xi = \frac{z_{12}z_{34}}{z_{13}z_{24}} = -4 \frac{y_I y_{II}}{|z_I - z_{II}|^2} , \quad (4.1)$$

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(and similar), before replacing  $\bar{z}$  by  $t$  in (3.3).



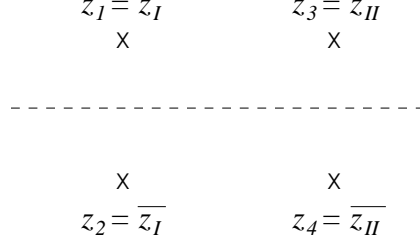


Fig. 1. Labeling of the points of the two-point function on the half plane with  $z_{ij} = z_i - z_j$  and  $z_{I/II} = x_{I/II} + iy_{I/II}$ . The “physical” cross ratio for the half plane is real negative.

#### 4.1 Neumann boundary condition on $\Phi$

In the Appendix, it is shown that for the two-point function of  $V_{\alpha_{1,2}}$ , the two-dimensional integral  $\oint d^2z(\cdots) = \frac{-i}{2} \int_{\partial M} dz \int^{z^*} dt(\cdots)$  from (3.3) reduces to one contour in each half plane between the two respective singularities. From the asymptotic behaviour, this double integral is seen to be the hypergeometric function

$$\begin{aligned}
I_1 &= \langle V_{\alpha_{1,2}}(z_I) V_{\alpha_{1,2}}(\bar{z}_I) V_{\alpha_{1,2}}(z_{II}) V_{\alpha_{1,2}}(\bar{z}_{II}) \rangle_{\mathcal{A}+\mathcal{A}_{\text{pert.}}}^{\text{Neumann}} \\
&= \frac{-i}{2} k \int_{z_1}^{z_3} ds \int_{z_2}^{z_4} dt \langle \cdots \rangle_{\mathcal{A}} \\
&= k_1 (z_{13} z_{24})^{\frac{2-M}{2(M+1)}} \left(1 - \frac{1}{\xi}\right)^{\frac{2-M}{2(M+1)}} {}_2F_1\left(\frac{2-M}{M+1}, \frac{1}{M+1}; \frac{2}{M+1}; \frac{1}{\xi}\right) \Big|_{M=3} \\
&= k_1 (z_{13} z_{24})^{-\frac{1}{8}} \left(1 - \frac{1}{\xi}\right)^{-\frac{1}{8}} \frac{1}{\sqrt{2}} \sqrt{1 - \frac{1}{\xi} + 1} , \tag{4.2}
\end{aligned}$$

where

$$k_1 = k B\left(\frac{1}{M+1}, \frac{1}{M+1}\right)^2 \Big|_{M=3} = k B\left(\frac{1}{4}, \frac{1}{4}\right)^2 , \tag{4.3}$$

with  $B$  being Euler’s Beta function. The expression in the second to last line is the general result for the two-point function of the vertex operator  $V_{1,2}$  in any minimal model labeled by  $M$ . For the determination of the asymptotic behaviour in the Appendix we used that the integration variables  $s$  and  $t$  stay in their respective half planes as illustrated in Figure 2a. The function  $I_1$  corresponds to the conformal block in which the insertions communicate in the  $\mathbb{I}$ -channel through the boundary.

Recall that, for the four-point function on the full plane [10], this is only one of the conformal block functions of the holomorphic sector, which consists of

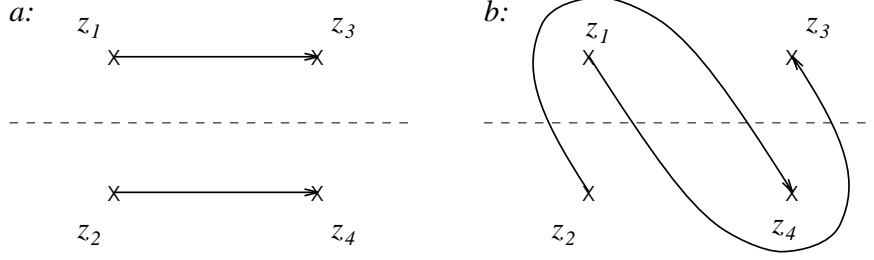


Fig. 2. The contours of two conformal blocks of the two-point function.

a: Neumann boundary condition on  $\Phi$ .

b: the other conformal block  $I_2$ .

any linear combination of (4.2) with

$$\begin{aligned}
 I_2 &= k_2 (z_{13} z_{24})^{\frac{2-M}{2(M+1)}} (1 - \frac{1}{\xi})^{\frac{2-M}{2(M+1)}} (-\frac{1}{\xi})^{\frac{M-1}{M+1}} {}_2F_1(\frac{1}{M+1}, \frac{M}{M+1}; \frac{2M}{M+1}; \frac{1}{\xi})|_{M=3} \\
 &= k_1 (z_{13} z_{24})^{-\frac{1}{8}} (1 - \frac{1}{\xi})^{-\frac{1}{8}} \frac{1}{\sqrt{2}} \sqrt{\sqrt{1 - \frac{1}{\xi}} - 1} ,
 \end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
 k_2 &= k \frac{M-2}{M-1} B(\frac{1}{M+1}, \frac{1}{M+1}) B(\frac{1}{M+1}, \frac{M-2}{M+1})|_{M=3} \\
 &= \frac{k}{2} B(\frac{1}{4}, \frac{1}{4})^2 \\
 &= \frac{1}{2} k_1 .
 \end{aligned} \tag{4.5}$$

The function  $I_2$  corresponds to the conformal block in which the insertions communicate in the  $\varepsilon$ -channel through the boundary.

The hypergeometric functions in (4.2) and in (4.4) have a branch cut from  $\xi = 0$  to  $\xi = 1$ . Crossing this branch cut changes the hypergeometric function  $I_1$  up to a phase  $e^{\frac{2\pi i}{8}}$  to  $I_2$ , and vice versa. This means that one obtains (4.4) from (4.2) by leaving the “physical” region (i.e. the negative real axis), analytically continuing over the branch cut and returning to the “physical” region. Therefore, the function (4.4) corresponds (up to the phase  $e^{\frac{2\pi i}{8}}$ ) to the contour integral in Figure 2b, in which the variables obviously leave their respective half planes.

It is important to observe that the two contours in Figure 2b still go from  $z_1$  to  $z_3$ , and from  $z_2$  to  $z_4$  respectively. What changes from one conformal block to the other, is the way these contours are twisted around the singular points, not the way the points are paired. The three different ways to pair up the four points correspond to the expansions around the three singular points of the cross ratios. An expansion around the zero of a certain cross ratio is analytic up to the prefactors of type  $z_{ij}^{\gamma_{i,j}}$  if the cross ratio tends to zero when bringing the two end points of one contour close to each other.

Hence, using the Neumann boundary condition, the two-dimensional screening integrals (3.1) lead to the conformal block  $I_1$ , while the block  $I_2$  a priori cannot be obtained by such an integral.

#### 4.2 The boundary conditions on the spin field $\sigma$

This subsection summarises the results of [11,17], and shows how they fit into the above description. Cardy [11] uses the facts that for  $|z_I - z_{II}| \rightarrow \infty$

$$\langle \sigma(z_I, \bar{z}_I) \sigma(z_{II}, \bar{z}_{II}) \rangle_{\text{h.p.}} \rightarrow \langle \sigma(z_I, \bar{z}_I) \rangle \langle \sigma(z_{II}, \bar{z}_{II}) \rangle , \quad (4.6)$$

and that for the free boundary condition  $\langle \sigma(z, \bar{z}) \rangle^{\text{free}} \equiv \langle \sigma(z) \sigma(\bar{z}) \rangle^{\text{free}} = 0$ , even in the limit  $y \rightarrow 0$ , i.e. for  $z \rightarrow \bar{z}$  at the boundary. This leads to the unique solution for the free boundary condition on  $\sigma$

$$\begin{aligned} \langle \sigma(z_I, \bar{z}_I) \sigma(z_{II}, \bar{z}_{II}) \rangle_{\text{h.p.}}^{\text{free}} &= (4y_I y_{II})^{-\frac{1}{8}} (1 - \xi)^{-\frac{1}{8}} \sqrt{\sqrt{1 - \xi} - 1} \\ &= \frac{1}{\sqrt{2}} (I_1 - I_2) , \end{aligned} \quad (4.7)$$

which is the difference between (4.2) and (4.4), and has a simple expansion around  $\xi = 0$ , corresponding to the  $\frac{1}{\xi}$ -expansion of (4.4)<sup>6</sup>.

On the other hand, in the limit of  $z_I$  and  $z_{II}$  close to each other and far away from the boundary, i.e. for  $\xi \rightarrow \infty$ , one can use the bulk operator product expansion which does not depend on the boundary condition<sup>7</sup>

$$\sigma(z_1, z_2) \sigma(z_3, z_4) \sim \frac{c_1 \mathbb{I}}{(z_{13} z_{24})^{\frac{1}{8}}} + c_2 (z_{13} z_{24})^{\frac{3}{8}} \varepsilon(z_3, z_4) + \dots . \quad (4.8)$$

Parametrising an arbitrary function in the space spanned by (4.2) and (4.4) as  $I = b_1 I_1 + b_2 I_2$ , one has therefore for both the free and fixed boundary conditions

$$\begin{aligned} b_1 &= c_1 = \frac{1}{\sqrt{2}} , \\ b_2 &= c_2 (-z_{12} z_{34})^{\frac{1}{2}} \langle \varepsilon(z_3, z_4) \rangle . \end{aligned} \quad (4.9)$$

From equation (4.7), it follows that

$$b_2^{\text{free}} = -\frac{1}{\sqrt{2}} . \quad (4.10)$$

<sup>6</sup> The cross ratio  $\xi$  in our paper is the inverse of the one used by Cardy [11].

<sup>7</sup> [13] used the corresponding argument for the  $\Phi_{1,3}$  operator in the  $O(N)$  model.

Using equation (4.9), this implies

$$c_2 (-z_{12}z_{34})^{\frac{1}{2}} = -\frac{1}{\sqrt{2} \langle \varepsilon(z_3, z_4) \rangle^{\text{free}}} . \quad (4.11)$$

In [17] it is pointed out that under the duality transformation interchanging the free and fixed boundary conditions, the energy operator  $\varepsilon$  changes to  $-\varepsilon$ . Therefore  $\langle \varepsilon(z, \bar{z}) \rangle^{\text{free}} = -\langle \varepsilon(z, \bar{z}) \rangle^{\text{fixed}}$ . Equation (4.9) leads for the fixed boundary condition on  $\sigma$  to

$$\begin{aligned} b_1^{\text{fixed}} &= c_1 = \frac{1}{\sqrt{2}} , \\ b_2^{\text{fixed}} &= c_2 (-z_{12}z_{34})^{\frac{1}{2}} \langle \varepsilon(z_3, z_4) \rangle^{\text{fixed}} = -\frac{\langle \varepsilon(z_3, z_4) \rangle^{\text{fixed}}}{\sqrt{2} \langle \varepsilon(z_3, z_4) \rangle^{\text{free}}} = +\frac{1}{\sqrt{2}} . \end{aligned} \quad (4.12)$$

Hence, for fixed boundary conditions, the conformal block is the sum of the two blocks  $I_1$  and  $I_2$ , and it has as well a clear analytic behaviour expanding around  $\xi = 0$ , corresponding to the  $\frac{1}{\xi}$ -expansion of (4.2).

Equations (4.12) and (4.7) being the sum, respectively the difference, of the two conformal blocks in which the  $\mathbb{I}$ - and  $\varepsilon$ -channel propagate through the boundary, fits very well to Cardy's relation between boundary states and boundary conditions (5.13) [12].

In terms of contour integrals, this means that the conformal blocks of the two-point function with free and fixed boundary conditions on  $\sigma$  are given by the integrals  $\int_{z_1}^{z_2} ds \int_{z_3}^{z_4} dt$ . For the former, the contours are twisted as shown in Figure 3a. For the later, they are straight as in Figure 3b. In both cases, one needs boundary-crossing contours which are not provided by the two-dimensional screening integral (3.1) using the Neumann boundary condition.

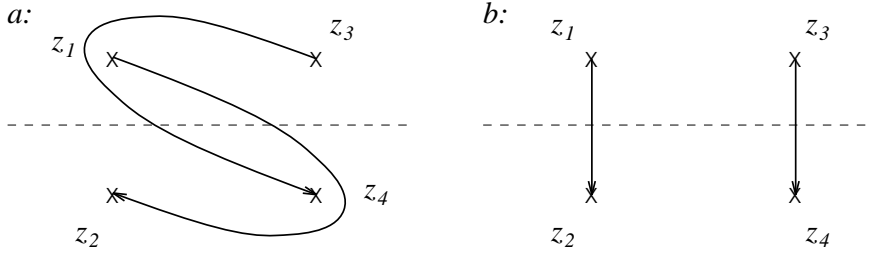


Fig. 3. The contours of two other conformal blocks of the two-point function.  
a: free boundary condition on  $\sigma$ . b: fixed boundary condition on  $\sigma$ .

A priori, it does not seem possible to use the manifestly conformally invariant formulation (3.1) and the Neumann condition (2.13), which is the natural boundary condition at the UV fixed point of the Coulomb gas, to describe the conformal boundary conditions of the minimal models we are interested in.

## 5 Boundary terms

### 5.1 Definition and usage

A solution to this dilemma is the insertion of one or more of the following boundary operators into the correlation functions

$$\begin{aligned} B_{1,2}(x_0) &:= \lim_{\delta \rightarrow 0} (2\delta)^{2\Delta_{1,2}} V_{\alpha_{1,2}}(x_0 + i\delta) V_{\alpha_{1,2}}(x_0 - i\delta) , \\ B_{2,1}(x_0) &:= \lim_{\delta \rightarrow 0} (2\delta)^{2\Delta_{2,1}} V_{\alpha_{2,1}}(x_0 + i\delta) V_{\alpha_{2,1}}(x_0 - i\delta) , \end{aligned} \quad (5.1)$$

and to define the correlation function for a particular boundary condition by

$$\begin{aligned} \langle X \rangle^{\text{b.c.}} &= f_0^{\text{b.c.}} \langle X \rangle^{\text{Neumann}} + f_1^{\text{b.c.}} \langle B_{1,2}(x_0) X \rangle^{\text{Neumann}} + \\ &+ f_2^{\text{b.c.}} \langle B_{1,2}(x_0) B_{1,2}(x_1) X \rangle^{\text{Neumann}} + \dots \end{aligned} \quad (5.2)$$

The point  $x_0$  is an arbitrary point on the boundary. The evaluation of correlators with these insertions is defined by first balancing the charges with the help of screening operators, then transforming the two-dimensional integrals into contour integrals, and finally taking the limit (or limites) of  $\delta \rightarrow 0$ . In the following, we will restrict our discussion to  $B_{1,2}$ . Equivalent statements are true for  $B_{2,1}$ .

Selecting a preferred point,  $x_0$ , (or even several  $x_i$ ) may seem unnatural, but this will correspond to the points at which a contour crosses the boundary, and the result will be independent of  $x_0$ . Observe as well that  $B_{1,2}$  has dimension 0. Suitably screened, it can be viewed as the product of two  $\sigma$  operators projected onto the identity. Thus in the full-plane description, the remaining operators will be trivial.

The insertion of a  $B_{1,2}$  generically causes correlation functions to vanish because of the limit  $\delta \rightarrow 0$ : the operator product expansion of the two vertex operators contributes with a leading factor  $\delta^{2\alpha_{1,2}^2}$ , the total exponent of  $\delta$  is therefore

$$2(\Delta_{1,2} + \alpha_{1,2}^2) > 0 . \quad (5.3)$$

However, the screening operator  $V_{\alpha_-}$  in the upper half plane has to encircle  $V_{\alpha_{1,2}}(x_0 + i\delta)$  on a Pochhammer contour which lies entirely in the upper half plane, while its mirror image has to go around  $V_{\alpha_{1,2}}(x_0 - i\delta)$  with a contour in the lower half plane. During their trip along the Pochhammer contour, these screening operators will pass momentarily, i.e. for a distance  $\sim 2\delta$ , through

a  $\delta$ -neighbourhood around  $x_0$ . The leading term of their operator product expansion with the other screeners and the two vertex operators in this neighbourhood will produce additional factors of  $\delta^{2\alpha_i\alpha_j}$ . Since the exponents of these factors are negative, we have to carefully calculate the total exponent before sending  $\delta \rightarrow 0$ .

If there are  $p$  screeners  $V_{\alpha_+}$  and  $q$  screeners  $V_{\alpha_-}$  in a  $\delta$ -neighbourhood of  $x_0$ , these factors are

$$\underbrace{\delta^{2\Delta_{1,2}} \delta^{2\alpha_{1,2}^2}}_{\text{from } B_{1,2}} \underbrace{\delta^{p+q}}_{\text{length of contribution}} \underbrace{\delta^{4(p\alpha_++q\alpha_-)\alpha_{1,2}}}_{\text{screeners with } B_{1,2}} \underbrace{\delta^{2\frac{p(p-1)}{2}\alpha_+^2} \delta^{2\frac{q(q-1)}{2}\alpha_-^2} \delta^{2pq\alpha_+\alpha_-}}_{\text{screeners with screeners}} . \quad (5.4)$$

The total exponent is

$$[p\alpha_+ + (q-1)\alpha_-]^2 \geq 0 , \quad (5.5)$$

and vanishes if and only if  $p = 0$  and  $q = 1$ . The corresponding result for the insertion  $B_{2,1}$  is:  $p = 1$ ,  $q = 0$ . Note that the general solution,  $p = rM$  and  $q = 1 + r(M+1)$ , only needs to be considered for  $r = 0$ : for BRST reasons, a collection of  $M$  screeners  $V_{\alpha_+}$  and  $M+1$  screeners  $V_{\alpha_-}$  vanishes when applied on any vertex operator [9].

Therefore, taking the limit  $\delta \rightarrow 0$  leaves only a non-vanishing contribution if exactly one of the screeners around the two vertex operators is trapped in the neighbourhood. The operator which remains at  $x_0$  is an uncharged identity operator. Having a contour coming from a point  $z$  in the upper half plane to  $x_0$  with only an identity operator at  $x_0$  and another contour from there to a point  $\bar{w}$  in the lower half plane, is however equivalent to a single contour joining  $z$  and  $\bar{w}$ :

$$\begin{aligned} & k \int_z^{x_0} ds \int_{\bar{w}}^{x_0} dt \langle B_{1,2}(x_0) V_{\alpha_-}(s) V_{\alpha_-}(t) X \rangle = \\ & = k \lim_{\delta \rightarrow 0} (2\delta)^{\frac{1}{8}} \int_z^{x_0+i\delta} ds \int_{\bar{w}}^{x_0-i\delta} dt \langle V_{\alpha_{1,2}}(x_0+i\delta) V_{\alpha_{1,2}}(x_0-i\delta) V_{\alpha_-}(s) V_{\alpha_-}(t) X \rangle \\ & = k \left\{ \int_z^{x_0} ds \langle \mathbb{I}(x_0) V_{\alpha_-}(s) X \rangle \cdot \right. \\ & \quad \cdot \lim_{\delta \rightarrow 0} \left( (2\delta)^{\frac{1}{8}} \int_{\bar{w}}^{x_0-i\delta} dt \langle V_{\alpha_{1,2}}(x_0+i\delta) V_{\alpha_{1,2}}(x_0-i\delta) V_{\alpha_-}(t) \rangle + \mathcal{O}(\delta) \right) + \\ & \quad \left. + \int_{\bar{w}}^{x_0} dt \langle \mathbb{I}(x_0) V_{\alpha_-}(t) X \rangle \cdot \right. \end{aligned}$$

$$\begin{aligned}
& \cdot \lim_{\delta \rightarrow 0} \left( (2\delta)^{\frac{1}{8}} \int_z^{x_0+i\delta} ds \langle V_{\alpha_{1,2}}(x_0+i\delta) V_{\alpha_{1,2}}(x_0-i\delta) V_{\alpha_-}(s) \rangle + \mathcal{O}(\delta) \right) \Big\} \\
& = i (-1)^{\frac{1}{4}} k \frac{1}{\sqrt{2}} B\left(\frac{1}{4}, \frac{1}{4}\right) \int_z^{\bar{w}} ds \langle V_{\alpha_-}(s) X \rangle .
\end{aligned} \tag{5.6}$$

In the last step, all powers of  $\delta$  (which cancel each other) were extracted, and it was used that the inner integral is an incomplete Beta function  $B_x(\frac{1}{4}, \frac{1}{4})$  at  $x = \frac{\bar{w}}{2i\delta} \rightarrow -\infty$  which produces the additional prefactors. Note that the final integral does obviously not depend on the point  $x_0$ .

Therefore, the insertions  $B_{1,2}$  and  $B_{2,1}$  produce boundary-crossing contours which were missing in the previous section. The need of such contours is not restricted to two-point functions. The conformal blocks without boundary-crossing contours are too restricted to be the blocks of  $N$ -point functions for generic conformal boundary conditions of the minimal model.

## 5.2 Generalisation to $B_{n,m}$

One can as well expect the appearance of generalised insertions

$$B_{n,m}(x_0) := \lim_{\delta \rightarrow 0} (2\delta)^{2\Delta_{n,m}} V_{\alpha_{n,m}}(x_0+i\delta) V_{\alpha_{n,m}}(x_0-i\delta) . \tag{5.7}$$

These operators need  $n-1$  screening operators  $V_{\alpha_+}$  and  $m-1$  screening operators  $V_{\alpha_-}$  encircling  $V_{\alpha_{n,m}}(x_0+i\delta)$  on Pochhammer contours which lie entirely in the upper half plane, and the same amount of screeners around  $V_{\alpha_{n,m}}(x_0-i\delta)$  with contours in the lower half plane. Similar to (5.4),  $p$  screeners  $V_{\alpha_+}$  and  $q$  screeners  $V_{\alpha_-}$  produce an exponent of  $\delta$

$$[(n-p-1)\alpha_+ + (m-q-1)\alpha_-]^2 \geq 0 , \tag{5.8}$$

what vanishes if and only if  $p = n-1$  and  $q = m-1$ . In the limit  $\delta \rightarrow 0$ , again only terms with exactly half of the screeners trapped in the neighbourhood will survive, which all leave an identity operator at  $x_0$ , together with several boundary crossing contours.

Although we will see that the  $B_{n,m}$  could be a quite useful link to Cardy's boundary states (see (5.13) or [12]), we wish to argue that the expressions  $B_{1,2}$  and  $B_{2,1}$  defined in (5.1) are the fundamental objects, and that, by combining them in the sense of quantum group representations [24], the general  $B_{n,m}$  are built up. One should however bear in mind that the expansion (5.2) could equally well be written as

$$\begin{aligned} \langle X \rangle^{\text{b.c.}} &= g_0^{\text{b.c.}} \langle X \rangle^{\text{Neumann}} + g_1^{\text{b.c.}} \langle B_{1,2}(x_0) X \rangle^{\text{Neumann}} + \\ &+ g_2^{\text{b.c.}} \langle B_{1,3}(x_0) X \rangle^{\text{Neumann}} + \dots \end{aligned} \quad (5.9)$$

As an example, we want to show how an insertion  $B_{1,2}(x_0)B_{1,2}(x_1)$  splits into a linear combination of  $B_{1,1}(x_0)$  and  $B_{1,3}(x_0)$ , if the limit  $x_1 \rightarrow x_0$  is taken before the limits  $\delta, \delta' \rightarrow 0$ . According to the discussion in section 4, it would seem natural that in a basis of conformal blocks, the contours in Figures 4a and 4b are the only contributions of  $x_0$  and  $x_1$ . However, the Dotsenko integrals between two singularities are special Pochhammer contours of trefoil type  $(1, 1, N - 2)$ . From the theory of multiple hypergeometric functions [25], it is known that these functions do not suffice to describe the general solution of the differential equation. For  $N \geq 4$ , one has as well to use contours of types  $(1, 2, N - 3)$ ,  $(2, 2, N - 4)$ , etc., which correspond to generalised Horn functions. This means that we have to consider contours as in Figure 4c, too.

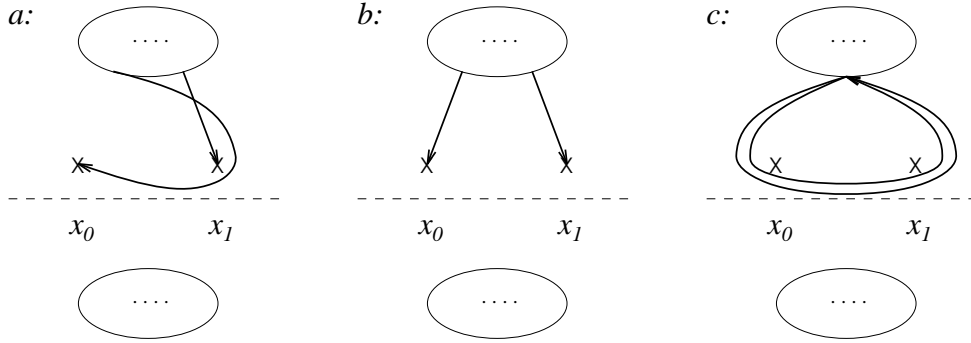


Fig. 4. Possible contours, the ellipses indicate other insertions

In the limit  $x_1 \rightarrow x_0$ , the contours of Figures 4a and 4b join to single contours and an identity operator as in (5.6). In Figure 4c however, the two  $V_{\alpha_{1,2}}$  vertex operators join to  $(x_1 - x_0)^{\frac{3}{8}} V_{\alpha_{1,3}}$ . Together with its mirror image and the factor  $\delta^{\frac{1}{8}} \delta'^{\frac{1}{8}}$ , this is the desired  $B_{1,3}$ . In the limit  $x_1 \rightarrow x_0$ , the insertion  $B_{1,2} B_{1,2}$  contributes therefore as a linear combination of  $B_{1,1}$  and  $B_{1,3}$ .

Equivalently, it is seen that a triple contour around three  $V_{1,2}$  operators at the boundary leaves a  $B_{1,4}$  operator in the limit  $x_1 \rightarrow x_0, x_2 \rightarrow x_0$ . In the Ising model, the  $V_{1,4}$  vertex operator is however  $V_{1,4} = (QV_{-2,0})$ , where  $Q$  is the BRST operator, and is hence trivial [9,24]. Thus, there is a truncation in the combination of  $B_{1,2}$  insertions, what strongly suggests that the  $B_{n,m}$  are the highest weight vectors of multiple tensor products of a representation of the quantum group  $SU_q(2)$  with itself. This is true for generic  $M$ , using  $q = e^{\frac{2\pi i}{2M}}$ . The contributing correlators are in the  $m = 0$  state, which is reached from the highest weight vector by application of an appropriate amount of screening (i.e. lowering) operators [24].

The truncation in the tensoring of the  $B_{1,2}$  operators is a consequence of the well-understood quantum-group structure of vertex operators. It is however



important for our situation, since it guarantees that there are only finitely many terms we have to sum over in (5.2) resp. (5.9) to get the conformal block of a generic conformal boundary condition. In the Ising model for example, there are only three terms contributing to get the conformal blocks of the free and fixed boundary conditions.

### 5.3 Boundary states and Virasoro eigenvalues

Denote the boundary state without any  $B_{1,2}$  insertions as  $|0, 0\rangle$ . The first zero stands for the  $L_0$ -eigenvalue

$$\int_{-\infty}^{\infty} dx T(x) |0, 0\rangle = 0 , \quad (5.10)$$

and the second zero is the  $SU_q(2)$  label. Then the insertion of one  $B_{1,2}$  (and appropriate screeners) gives a state  $|1\rangle = \sum_m c_m |\frac{1}{16}, m\rangle$ . The insertion of two  $B_{1,2}$  leads with the help of the representation theory of the quantum group  $SU_q(2)$  to a linear combination of states  $|2\rangle = d|0, 0\rangle + \sum d_m |\frac{1}{2}, m\rangle$ . Insertions of any odd (even) number of  $B_{1,2}$  give linear combinations of the form  $|1\rangle$  (resp.  $|2\rangle$ ), with different coefficients. Note that in the quantum group interpretation of the last subsection, the  $B_{n,m}$  are highest weight vectors which still have to be lowered by screening operators.

Using (5.10) and splitting the contour into the vanishing integral on the real axis and a circle around  $x_0 + i\delta$ , resp.  $x_0 - i\delta$ , one immediately can verify that the first label is both the  $L_0$  and  $\bar{L}_0$  eigenvalue.

### 5.4 Discussion for the Ising model

In the next two subsections, we want to investigate the insertion of  $B_{1,2}$  terms in two examples. Especially, we would like to see how much information we can gain on the coefficients  $f_i^{\text{b.c.}}$  in (5.2).

From the remarks on truncation at the end of subsection 5.2, we already know that for the Ising model only the first three of the  $f_i$  can be non-vanishing. Observe that for an odd number  $N$  the  $N$ -point function of the  $\sigma$  operator vanishes for the free boundary condition:

$$\langle \sigma(z, \bar{z}) \rangle^{\text{free}} = 0 , \quad \langle \sigma(z_1, \bar{z}_1) \sigma(z_2, \bar{z}_2) \sigma(z_3, \bar{z}_3) \rangle^{\text{free}} = 0 , \quad \dots \quad (5.11)$$

This can be obtained in the Coulomb-gas picture by requiring that the free

boundary condition for  $\sigma$  corresponds to a linear combination of terms with even numbers of  $B_{1,2}$  insertions only (i.e.  $f_1^{\text{free}} = 0$ ). For an odd number of  $V_{\alpha_{1,2}}$  in the upper half plane, the charges cannot be balanced by the insertion of screeners, and all odd-point functions of the  $V_{\alpha_{1,2}}$  operator, resp.  $\sigma$ , vanish. As it should be, the  $\mathbb{Z}_2$  degeneracy for the free boundary condition appears as well: all non-vanishing correlators are invariant under  $\sigma \rightarrow -\sigma$ .

The fixed boundary condition, on the other hand, requires a contribution of a single  $B_{1,2}$  insertion for the one-point function of  $\sigma$ . It should carry the sign of the external magnetic field, which in the conformal limit is  $h \rightarrow \pm\infty$ . Since the charges in the other to terms in (5.2) can not be balanced, the one-point function would then read

$$\begin{aligned}
\langle \sigma \rangle^{\text{fixed}} &= \\
&= f_1^{\text{fixed}} \lim_{\delta \rightarrow 0} (2\delta)^{\frac{1}{8}} \langle V_{\alpha_{1,2}}(x_0 + i\delta) V_{\alpha_{1,2}}(x_0 - i\delta) V_{\alpha_{1,2}}(z) V_{\alpha_{1,2}}(\bar{z}) \rangle_{\mathcal{A}_{\text{pert}}} \\
&= k f_1^{\text{fixed}} \lim_{\delta \rightarrow 0} (2\delta)^{\frac{1}{8}} \int_{x_0 + i\delta}^z dw \int_{x_0 - i\delta}^{\bar{z}} d\bar{w} \cdot \\
&\quad \cdot \langle V_{\alpha_-}(w) V_{\alpha_-}(\bar{w}) V_{\alpha_{1,2}}(x_0 + i\delta) V_{\alpha_{1,2}}(x_0 - i\delta) V_{\alpha_{1,2}}(z) V_{\alpha_{1,2}}(\bar{z}) \rangle^{\text{Neumann}} \\
&= \frac{1}{\sqrt{2}} B(\frac{1}{4}, \frac{1}{4}) k f_1^{\text{fixed}} \cdot \int_z^{\bar{z}} dw \langle V_{\alpha_-}(w) V_{\alpha_{1,2}}(z) V_{\alpha_{1,2}}(\bar{z}) \rangle \\
&= \frac{1}{\sqrt{2}} B(\frac{1}{4}, \frac{1}{4})^2 k f_1^{\text{fixed}} \cdot (z - \bar{z})^{-\frac{1}{8}}, \tag{5.12}
\end{aligned}$$

where we have used (5.6). By comparing this result with [17] one gets for the coefficients  $k f_1^{\text{fixed}} B(\frac{1}{4}, \frac{1}{4})^2 = 2^{\frac{3}{4}}$ . A single  $B_{1,2}$  can however not be the full story for the fixed boundary condition. If there were no contribution of an insertion of a pair of  $B_{1,2}$  or of the Neumann boundary condition, the two-point function of  $\sigma$  would vanish. Recalling the conformal blocks (4.7) and (4.12), as well as the corresponding Figures 3a and 3b, one knows that there has to be a contribution of  $B_{1,2} B_{1,2}$  for free and fixed boundary conditions. But a correlator with four insertions in each half plane has five independent cross ratios and 32 conformal blocks. This makes it quite difficult to use the techniques described in the Appendix to determine with which conformal block one ended up. However, there is an easier way of gaining further insight:

Consider the one-point function of  $\varepsilon$  in the expansion (5.9) for which as well only the first three terms contribute. Only for the  $B_{1,3}$ -insertion, the charges can be balanced, and the other two terms do not contribute. From the sign change under the duality transform, one gets  $g_2^{\text{fixed}} = -g_2^{\text{free}}$ . Now recall that equations (4.12) and (4.7) are the sum, respectively the difference, of the two conformal blocks in which the  $\mathbb{I}$ - and  $\varepsilon$ -channel propagate through the boundary. Hence,  $g_0^{\text{fixed}}$  is  $g_0^{\text{free}}$ . With the help of the quantum group Clebsch-

Gordon coefficients the  $g_i$  can be transformed into  $f_i$ .

This can be compared to Cardy's result in [12]. He introduces boundary states  $|j\rangle = \sum |j, N\rangle \otimes U|j, N\rangle$  which fulfill all the boundary conditions required by the underlying  $W$ -algebra and are  $W$ -descendents of the highest weight state  $|j, 0\rangle$  with eigenvalue  $j$  under the action of  $L_0 \equiv \bar{L}_0$ <sup>8</sup>.

Cardy identifies the states corresponding to the conformal boundary conditions of the Ising model as

$$\begin{aligned} |f\rangle &= |0\rangle - |\varepsilon\rangle, \\ |\pm\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|\varepsilon\rangle \pm \frac{1}{\sqrt{2}}|\sigma\rangle. \end{aligned} \quad (5.13)$$

Our results hence suggest that we should interpret (5.13) such that we have to subtract the conformal block without boundary insertions and the one with a  $B_{1,3}$ -insertion from each other to get the conformal block of the free boundary condition. The fixed boundary condition equivalently would be given as a linear combination of no insertion, a  $B_{1,2}$ -insertion and a  $B_{1,3}$ -insertion.

In this interpretation, the odd-point functions of the  $\sigma$  operator vanish for free boundary conditions. The one-point function carries the sign of the external magnetic for the fixed boundary conditions. The sign change in front of the  $I_2$ -term in equations (4.7) and (4.12) is explained, as well the sign change under the duality transform for  $\langle\varepsilon\rangle$ <sup>9</sup> [17].

The similarity between (5.2) and (5.9) with the coefficients just derived, and (5.13) has to be used with caution, though. Cardy's states are an infinite sum over  $W$ -descendents, including the Virasoro descendents which have shifted  $L_0$  eigenvalues, and are furthermore not normalisable.

### 5.5 A further example: the 3-state Potts model

For the 3-state Potts model (the minimal Model at  $M = 5$ ), there are branch cuts of order five for a screener going around an insertion point. For this example, we do not have an as rigorous proof as for the Ising model that the

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<sup>8</sup> These states should not be confused with Cardy's  $\widetilde{|j\rangle}$ : along an infinitely long strip only  $\Phi_j$  propagates for the boundary states  $\langle 0|$  and  $\widetilde{|j\rangle}$ . Cardy maps the strip later to a half plane, introducing boundary-condition changing operators at the origin. Our boundary is equivalent to one side of the strip, the other one being moved to an infinite distance, we will not consider boundary-condition changing operators in this paper.

<sup>9</sup> The factor of  $\sqrt{2}$  could be explained by a different normalisation in (5.13).

two-point function for the Neumann boundary condition is the conformal block given by the double integral of Figure 2a<sup>10</sup>. This means especially that we cannot say anything about the relative coefficient in front of the two versions of the integral, but that it is given by a sum of phase factors of order five.

Consider the  $\varepsilon$  operator (with conformal weight  $\Delta_{2,1} = \frac{2}{5}$ ). Its two-point function for the Neumann boundary condition is given by the hypergeometric function

$$\langle \varepsilon \varepsilon \rangle^{\text{Neumann}} = c \cdot (z_{13} z_{24})^{-\frac{4}{5}} (1 - \frac{1}{\xi})^{-\frac{4}{5}} {}_2F_1(-\frac{8}{5}, -\frac{1}{5}; -\frac{2}{5}; \frac{1}{\xi}) . \quad (5.14)$$

The free boundary condition of the 3-state Potts model is according to [12] given by the sum of  $|0\rangle$  and  $|\varepsilon\rangle$ . For the former however, we can not balance the charges, and it has to vanish. Hence, the one-point function is given as the conformal block given by a  $V_{2,1}$  insertion with a  $B_{2,1}$  at the boundary. We get

$$\langle \varepsilon \rangle^{\text{free}} = \text{const} \cdot (2y)^{-\frac{4}{5}} , \quad (5.15)$$

what is in accordance with [17].

## 6 Outlook

It would be a natural expectation that the boundary terms (5.1) can be derived by adding boundary terms to the action. Changing the action to

$$\begin{aligned} \mathcal{A}_{\text{free}} &= \mathcal{A} + B_{1,3} , \\ \mathcal{A}_{\text{fixed}} &= \mathcal{A} \pm B_{1,2} \end{aligned} \quad (6.1)$$

would produce many of the qualitative features required above, as e.g. the vanishing of the odd-point functions or the sign of the  $\varepsilon$  one-point function. An exact calculation of the appearing coefficients is however cumbersome, and is beyond the scope of this paper. One of the difficulties lies in the fact that beside the factor  $k$  in front of the screening term in the action, there are now additional free parameters in front of the  $B_{n,m}$  in (6.1).

By a series expansion of the exponential  $e^{B_{1,2}(x_0)}$ , with multiple points  $x_i$  for the higher order terms, the changed action leads to an expansion of multiple insertions of  $B_{1,2}$ . In such an expansion, one gets arbitrarily many  $B_{1,2}$  insertions. However, once there are more  $B_{1,2}$  than other insertions, one is forced

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<sup>10</sup> Nevertheless, we still think that the arguments given outside the proof are strong enough for this to be true.

to put screening contours around pairs of  $V_{1,2}$  operators close to the boundary. But such terms would as well appear in the partition function, and one can use standard arguments of quantum field theory to show that only the “connected” pieces contribute. As shown above, these are only finitely many terms.

One can compare (6.1) to the boundary integrals in [1] where the off-critical Ising model is considered as an example. The critical Ising model we considered so far is the massless limit of the bosonisation of the free Majorana fermion field theory

$$S = \iint_M d^2z (\Psi \bar{\partial} \Psi - \bar{\Psi} \partial \bar{\Psi} + m \Psi \bar{\Psi}) . \quad (6.2)$$

If one considers the off-critical theory with free boundary condition on the half plane, one has to add a boundary term [1]

$$S_{\text{h.p.}}^{\text{free}} = S + \frac{1}{2} \int_{\mathbf{R}} dx (\Psi \bar{\Psi} + a \partial_x a) . \quad (6.3)$$

Ghoshal and Zamolodchikov introduce  $a$  as a fermionic boundary field. The  $\Psi \bar{\Psi}$  term is an  $\varepsilon(z, \bar{z})$  in the critical theory. This corresponds to the suggested  $B_{1,3}$  in (6.1).

The coupling to the external magnetic field  $S_h = S^{\text{free}} + h \int dx \sigma_B(x)$ , where  $\sigma_B = \frac{1}{2}(\Psi + \bar{\Psi})a$ , disappears under the limit  $m \rightarrow 0$  unless  $h \rightarrow \pm\infty$ , for which it leads to the fixed boundary conditions. The boundary spin operator  $\sigma_B$  can easily be related to the pair of  $V_{\alpha_{1,2}}$  operators arising from  $B_{1,2}$  at the boundary.

The boundary terms (6.3) are still present after scaling the theory to the conformal point. Rewritten via a one dimensional Stoke’s theorem – the points where the branch cuts cross the real axis forming the “boundary” of the real axis – they might explain the origin of  $B_{1,2}$  and  $B_{2,1}$ . It even leaves the option to relate them to the additional degree of freedom  $a$  [1] which, for scaling reasons, must have dimension 0 like  $B_{1,2}$ . The introduction of boundary spin operators was considered for more general conformal models in [2,26].

Additions of boundary terms to the action, which are similarly to the terms in (6.1) no integrals, appear as well for Toda theories [15,16]. It remains unclear whether the  $B_{n,m}$  can be related to these boundary terms (recall that the Coulomb gas considered here is the  $A_1$  Toda theory). The boundary term in [15,16] produces for the ordinary Lie algebra  $A_1$  the “screeners” of the right hand side of (2.14). It would be nice to interpret this as the second conformal block (4.4), which has to be added to the Neumann term (4.2) to get

the conformal blocks of free and fixed boundary conditions. Since the higher minimal models ( $M > 3$ ) have more than two conformal boundary conditions, but are still described by  $A_1$ , this interpretation is however unlikely to work.

## 7 Conclusion

The Neumann boundary condition on  $\Phi$  is the natural boundary condition for a generic non-affine Toda field theory viewed as the UV limit of an affine theory. Treating the Coulomb-gas description of minimal models as a perturbation of the free field theory by a Liouville potential, the Neumann boundary condition leads to screening contours which do not cross the boundary. On the other side, the  $N$ -point functions for conformal boundary conditions of the minimal model fall into conformal blocks which generically correspond to boundary-crossing contours.

We introduced boundary insertions  $B_{1,2}$  and  $B_{2,1}$  (5.1) which can be described as composite operators of a vertex operator and its mirror image. These insertions sew together contours from each half plane to a boundary-crossing contour. Only an identity operator remains at the point where this contour crosses the boundary, while all other contributions vanish. It is argued that these insertions can combine to general  $B_{n,m}$ . The appearing truncation follows from the quantum group behaviour of the vertex operators, with  $q = e^{\frac{2\pi i}{M}}$ .

Although the calculational difficulty is increased by the additional insertions, sensitive statements can be made in simple situations. There is a close connection between Cardy's boundary states (5.13) and the coefficients in (5.2).

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## A Derivation of the contour integrals from the two-dimensional integral

### A.1 The differential equation

Consider the  $\sigma$ -operator two-point function of the Ising model on the half plane. Recall that this correlator, written as an analytic four-point function has to fulfill the differential equation [27]

$$D_1 I = \left[ \frac{4}{3} \partial_1^2 - \frac{\partial_2}{z_{12}} - \frac{\partial_3}{z_{13}} - \frac{\partial_4}{z_{14}} - \frac{1}{16} \left( \frac{1}{z_{12}^2} + \frac{1}{z_{13}^2} + \frac{1}{z_{14}^2} \right) \right] I = 0 \quad (\text{A.1})$$

and three similar equations with  $z_2$ ,  $z_3$  and  $z_4$  taking the special role. Plugging the integral representation (3.3) into this differential equation one gets

$$D_1 \iint ds dt (\dots) = \iint ds dt \left( \partial_s \frac{(\dots)}{z_1 - s} + \partial_t \frac{(\dots)}{z_1 - t} \right), \quad (\text{A.2})$$

what obviously vanishes applying Stoke's theorem, if the  $s$ - and the  $t$ -integrations are along closed (i.e. Pochhammer) contours. Integrals between two pairs of singularities, plugged into (A.2), leave divergent expressions at the two end points, which are not easily seen to cancel each other. However, a Dotsenko integral between two singularities is up to a prefactor a Pochhammer integral around the same pair of points<sup>11</sup>. Therefore, double integrals with both contours going from one singularity to another fulfill (A.1), too.

For the area integral  $\iint d^2 z (\dots)$ , Stoke's theorem applied to (A.2) leads to integrals along the branch cuts which vanish since the integration along the left and right sides of the cuts cancel each other because of monodromy invariance of the integrand. For the half plane, there are as well integrals along the real axis, but these vanish due to the factor  $(z - z^*)^{\frac{3}{2}}$ . Therefore, the area integrals fulfill the differential equations both for the full and the half plane, and hence they are a linear combination of the conformal blocks.

### A.2 Evaluation of the two-dimensional integral

We argued in section 4 that it is a priori impossible to get boundary crossing contours by evaluating the two-dimensional integral  $\iint d^2 z (\dots)$ . A screener and its mirror image are on two different half planes, hence after applying Stoke's theorem as in (3.3) one integration is performed along the boundary of one half plane (the boundary includes the branch cuts) while the other integration

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<sup>11</sup> It is easy to see that the branch cuts give weights  $1 + q + 1 - q^{-1} = 2 - 2\text{Re } q$  to the four parts of the Pochhammer contours, and that the contributions in an  $\delta$ -neighbourhood of the singularities are proportional to  $\delta \cdot \delta^{-\frac{3}{4}} \sim \delta^{\frac{1}{4}} \rightarrow 0$ .

goes from a reference point <sup>12</sup> in the other half plane to the complex conjugate of the first integration variable. Therefore each integration variable stays in its respective half plane. Since there are no distinguished points on the boundary, the contours have to run between the singular points in the bulk.

To show for a certain example whether this argumentation is indeed correct, one can use the following tools:

a) One chooses a boundary of the upper half plane by carefully defining the branch cuts of  $(z_i - s)^{2\alpha - \alpha_{n,m}}$ . The branch cuts for  $(\bar{z}_i - t)^{2\alpha - \alpha_{n,m}}$  then have to be the mirror image of the former cuts.

b) The boundary can then be cut into pieces between two singularities, pieces going to the boundary and boundary pieces <sup>13</sup>.

c) The  $J$ -terms, which are in the upper half plane along one of these pieces and in the lower half plane from one end point to the complex conjugate of the first integration variable, appear in pairs which cancel each other. The  $J$ -terms along the boundary vanish for reality reasons (see  $f$ ) and  $g$ )).

d) Integrals along closed loops not encircling any singular point vanish if and only if the double integral is not a  $J$ -term.

e) Carefully calculating the monodromy coefficients, one moves all pieces to one side of the branch cuts.

f) Since vertex operators and their mirror images have the same charges, the complex conjugate of a double integral is obtained by integrating between the complex conjugates of the end points (paying careful attention to the branch cuts).

g) From the way we defined the integrand after (3.5), it is clear that integrals  $\iint d^2z (\dots)$  are real. Application of Stoke's theorem via (3.3) yields an  $\frac{-i}{2}$ , hence the sum of the double integrals has to be purely imaginary. We can subtract the complex conjugate of all these terms and divide by 2.

For the two-point function of the  $\sigma$  operator in the Ising model (and as well

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<sup>12</sup> If one chose the reference point in the same half plane as the first integration variable, one would not get a contradiction to this statement. The integral from this reference point to the complex conjugate of the first integration variable can always be performed via a new reference point in the other half plane. It hence splits into the integral which we consider and an integral with fixed end points. The later can be moved out of the first integral, which then vanishes for homotopy reasons.

<sup>13</sup> The integrals along the half circle at infinity is easily shown to vanish. In the compactified picture of a half sphere instead of the half plane, this means just that there is nothing special about the point  $z = \infty$ .



for the  $\Phi_{2,1}$  operator of the tri-critical Ising model  $M = 4$ ), the integrals going to the boundary have vanishing overall factor, and one remains with  $\iint d^2z (\dots) = 1 \cdot \int_{z_1}^{z_3} ds \int_{z_2}^{z_4} dt (\dots)$ . This proof does not work as straightforward for other examples, where one has to use nontrivial relations between different integrals going to the boundary, and it is hence far from being general<sup>14</sup>. Especially, there might be a proportionality factor different from 1.

From the remarks made at the beginning of this subsection, one furthermore expects that the contours are symmetric under reflection along the real axis, a symmetry which ensures monodromy invariance and reality. This statement is obviously fulfilled for the case considered above. It is true in general since the two-dimensional integral has to yield a function which depends on the *real* cross ratios only.

### A.3 Asymptotic behaviour

For large  $R = z_{12} \approx z_{34} \mathbf{g} z_{13} \approx z_{24}$ , the asymptotic behaviour of the integral  $\int_{z_1}^{z_3} ds \int_{z_2}^{z_4} dt (\dots)$  gives after the substitutions  $s = z_1 - z_{13}u$  and  $t = z_2 - z_{24}v$  two separate integrals for  $u$  and  $v$  in  $\mathcal{O}(R^0)$ , which are easily identified as Euler's Beta-functions.  $\mathcal{O}(R^{-1})$  vanishes. The subleading order is  $\mathcal{O}(R^{-2})$  what leads with  $\frac{1}{\xi} \sim \frac{z_{13}z_{24}}{R^2}$  to

$$I_{\text{Neumann}} \sim (z_{13}z_{24})^{-\frac{1}{8}} (B(\frac{1}{4}, \frac{1}{4})^2 R^0 + \mathcal{O}(\frac{1}{\xi})) . \quad (\text{A.3})$$

Therefore, there is no subleading contribution of  $\mathcal{O}((z_{13}z_{24})^{\frac{3}{8}})$ , i.e. no contribution of (4.4) and the integral is identified as (4.2).

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<sup>14</sup> Even if one argued that there might be an exotic example for which the two-dimensional integral might give a conformal block which corresponds to boundary-crossing contours, this would be *one* well-defined block and to derive the others one needs to introduce the  $B_{n,m}$ .

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